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ELASTOPLASTIC WAVES IN GRANULAR MATERIALS

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For a description of the deformation of materials with different resistances to tension and compression, the conventional rheological diagram is supplemented by a new element — rigid contact. It is used to construct a model of an ideal granular material possessing elastic and plastic properties. The loose state of the material is described by the Mises–Schleicher strength conditions, and transition to the plastic state is described by the Mises yield condition. The model proposed is employed to study the propagation of longitudinal elastic and plastic compression shock waves. It is shown that one- or two-wave configurations of discontinuities occur depending on the rate of compression and the degree of the initial loosening of the material.

Key words: *granular material, elasticity, plasticity, shock adiabat, variation inequality.*

Introduction. The theory of granular materials is an actively developing field of mechanics. Although one of the first papers on this topic [1] was published almost a hundred years ago, mathematical models of such materials are far from being completed. At present, there are two classes of models corresponding to two regimes: the regime of quasistatic deformation and the regime of fast motion. Models of the first class describe the behavior of closely packed media using the theory of plastic flow with the Coulomb–Mohr or Mises–Schleicher boundary conditions. Models of the second class treat loose media (ensembles of large numbers of interacting particles) from the viewpoint of the kinetic theory of gases.

An overview of papers on the mechanics of fast motions of granular materials is given in [2]. For quasistatic deformation, stress theory in plane, statically definable problems has been developed and has been widely used in soil mechanics [3]. In such problems, kinematic characteristics are determined from an associated flow law [4]. In [5], a nonassociated law is proposed that provides a more exact description of the velocity field upon intrusion of a rigid punch in sand. A shortage of these approaches lies in the fact that in rigid unloading, the strain rate tensor is equal to zero, and, therefore, after loosening, the material cannot be compressed. Thus, the kinematic laws are applicable only in the case of monotonic loading.

The equations of uniaxial dynamic deformation of an ideal granular material with elastic properties are studied in [6]. It is shown that along with velocity discontinuities (shock waves), they also describe displacement discontinuities. In [7], these equations are used to analyze the "dry boiling" process — spontaneous occurrence and collapse of discontinuities in the bulk of a material. Phenomenological models of a spatial stress-strained state of cohesive soils are presented in [8, 9].

In the present study, we propose a geometrically linear model for the spatial deformation of an elastoplastic granular material. This model is employed to study the distribution of longitudinal compression shock waves. The mechanical parameters of the model are specified differently from [10].

1. Mathematical Model. For a phenomenological description of materials with different resistances to tension and compression, the conventional rheological diagrams is supplemented by a new element — rigid contact (Fig. 1a). For compressing stresses, this element is not deformed. For zeroth stress, the strain can be an arbitrary positive quantity. Tensile stresses are inadmissible. The diagrams in Fig. 1b and c correspond to the models of elastic and elastoplastic granular media. In compression, such media are in an elastic state or a plastic state, and in tension, the stresses are equal to zero.

Combining rigid contact with elastic, plastic and viscous elements, one can construct rheological models for more complicated media.

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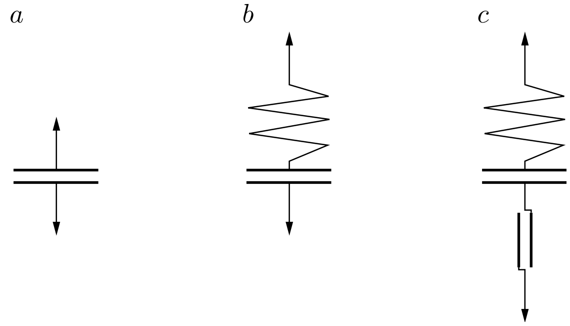

Fig. 1. Rheological diagrams of granular materials: (a) rigid granules; (b) elastic granules; (c) elastoplastic granules.

The present mathematical model of rigid contact (ideal granular material with rigid granules) reduces to the system of relations

$$\sigma \leqslant 0, \qquad \varepsilon \geqslant 0, \qquad \sigma\varepsilon = 0,$$

where σ is the stress and ε is the strain. According to this model, in uniaxial deformation of the material, only two states are possible: compression, where $\sigma < 0$ and $\varepsilon = 0$, and loosening, where $\varepsilon > 0$ and $\sigma = 0$.

The model can also be written as the variational inequalities

$$\sigma(\tilde{\varepsilon} - \varepsilon) \leqslant 0, \quad \varepsilon \geqslant 0, \quad \tilde{\varepsilon} \geqslant 0, \qquad (\tilde{\sigma} - \sigma)\varepsilon \leqslant 0, \quad \sigma \leqslant 0, \quad \tilde{\sigma} \leqslant 0$$

($\tilde{\sigma}$ and $\tilde{\varepsilon}$ are arbitrarily varied quantities) each of which admits the potential representation

$$\sigma \in \partial\varphi(\varepsilon), \qquad \varepsilon \in \partial\psi(\sigma). \tag{1.1}$$

Here the stress and strain potentials φ and ψ are indicator functions, which are equal to zero on the cones $C = \{\varepsilon \geqslant 0\}$ and $K = \{\sigma \leqslant 0\}$, respectively, and are equal to infinity outside these cones. Below, these functions are denoted by $\delta_C(\varepsilon)$ and $\delta_K(\sigma)$. The symbol ∂ is used to denote the subdifferential

$$\partial\varphi(\varepsilon) = \{\sigma \mid \varphi(\tilde{\varepsilon}) - \varphi(\varepsilon) \geqslant \sigma(\tilde{\varepsilon} - \varepsilon) \quad \forall\, \tilde{\varepsilon}\},$$

which is a set of angular coefficients of linear functions, whose plots pass through the point $(\varepsilon, \varphi(\varepsilon))$ and lie below the plot of the function φ.

In the present paper, the use of the notion of a subdifferential, which is a generalization of the notion of a derivative, is due to the fact that the potentials introduced are not differentiable functions. The same situation is typical of models of plasticity theory, in which an associated flow law is formulated in terms of subdifferentials [11]. However, unlike in plasticity theory, in the case considered, relations (1.1) represent a nonlinear Hooke's law and, thus, describe a nondissipative deformation mechanism.

The rigid contact model is extended to the case of a spatial stress-strained state using inclusions (1.1). For this, it is necessary to specify a convex cone C in the space of strain tensors or a cone K in the space of stress tensors. If one of the cones is known, the second is found as the conjugate:

$$K = \{\sigma \mid \sigma : \varepsilon \leqslant 0 \quad \forall\, \varepsilon \in C\}, \qquad C = \{\varepsilon \mid \sigma : \varepsilon \leqslant 0 \quad \forall\, \sigma \in K\}$$

(colon denotes convolution of the tensors). The corresponding potentials (indicator functions of the cones C and K) are dual; i.e., they are determined from one another by Young's transformation:

$$\varphi(\varepsilon) = \sup_{\sigma}\{\sigma : \varepsilon - \psi(\sigma)\}, \qquad \psi(\sigma) = \sup_{\varepsilon}\{\sigma : \varepsilon - \varphi(\varepsilon)\}.$$

Available experimental data on the deformation properties of hard sands confirm the hypothesis on the elastic state of the material at stresses close to hydrostatic compression. Such stresses are internal points of the cone K. For an elastic granular material (Fig. 1b), $\psi(\sigma) = \sigma : a : \sigma/2 + \delta_K(\sigma)$ (a is the tensor of the elastic compliance modulus of the fourth rank that corresponds to the model of an elastic element). The constitutive relations (1.1) are reduced to the Haar and Kármán inequality [1]

$$(\tilde{\sigma} - \sigma) : (a : \sigma - \varepsilon) \geqslant 0, \qquad \sigma, \tilde{\sigma} \in K. \tag{1.2}$$

Taking into account the symmetry and positive definability of the tensor a, it is possible to show that a solution of inequality (1.2) is the stress tensor $\sigma = s^{\pi}$ that is equal to the projection of the conditional stress tensor s determined from the linear Hooke's law $a : s = \varepsilon$ onto K along the norm $|\sigma|_a = \sqrt{\sigma : a : \sigma}$.

We assume that the cone K is specified in the form $f_i(\sigma) \leqslant 0$ $(i = 1, \ldots, m)$, where f_i are convex differentiable functions. Then, under the Kuhn–Tacker theorem [12], the problem of determining the projection is equivalent to the problem of determining a Lagrangian saddle point

$$L(\sigma, \lambda) = \frac{1}{2}\,|\sigma - s|_a^2 + \sum_{i=1}^{m} \lambda_i f_i(\sigma), \qquad \lambda_i \geqslant 0.$$

In this case, the following system of equations is satisfied:

$$a : (\sigma - s) + \sum_{i=1}^{m} \lambda_i \,\frac{\partial f_i(\sigma)}{\partial \sigma} = 0, \qquad \lambda_i f_i(\sigma) = 0. \tag{1.3}$$

For an elastoplastic granular material (Fig. 1c), the strain tensor is decomposed into a sum of elastic and plastic components: $\varepsilon = \varepsilon^e + \varepsilon^p$. The elastic strain tensor obeys inequality (1.2), which allows for the possible loosening of the material. The plastic strain rate tensor satisfies the constitutive relations of flow theory [13]:

$$\sigma \in \partial\eta(\dot\varepsilon^p). \tag{1.4}$$

Here η is the dissipative stress potential which is a positive uniform convex function of strain rates. The potential is homogenous because the plastic deformation process does not depend on the time scale. By virtue of this property, the dual potential $\chi(\sigma)$ [Young's transformation of the function $\eta(\dot\varepsilon)$] is equal to the indicator function of the convex set

$$F = \{\sigma \mid \sigma : \dot\varepsilon \leqslant \eta(\dot\varepsilon) \quad \forall\, \dot\varepsilon\}.$$

The boundary F in the space of stresses is the yield surface of the material.

Relations (1.4) are written in the equivalent form $\dot\varepsilon^p \in \partial\chi(\sigma)$, leading to the Mises inequality

$$(\tilde\sigma - \sigma) : \dot\varepsilon^p \leqslant 0, \qquad \sigma, \tilde\sigma \in F. \tag{1.5}$$

If the set F is a cylinder with a hydrostatic stress axis, the bulk deformation of the material obeys a linear elastic law. Otherwise, the model describes irreversible bulk compression of the material. Generally, the set F is parametrized as $g_j(\sigma) \leqslant 0$ $(j = 1, \ldots, n)$, where g_j are differentiable convex functions, and the associated flow law

$$\dot\varepsilon^p = \sum_{j=1}^{n} \lambda_j \,\frac{\partial g_j(\sigma)}{\partial \sigma}, \qquad \lambda_j g_j(\sigma) = 0 \qquad (\lambda_j \geqslant 0)$$

obtained from (1.5) using the Kuhn–Tacker theorem is satisfied.

Inequality (1.2) for ε^e and inequality (1.5) together with the equations of motion and the kinematic equations

$$\rho\dot{\boldsymbol v} = \nabla \cdot \sigma, \qquad 2\dot\varepsilon = \nabla \boldsymbol v + (\nabla \boldsymbol v)^*$$

form a closed model that describes the dynamics of a granular material. Here ρ is the density, $\boldsymbol v$ is the velocity vector, and ∇ is the gradient vector; the asterisk denotes transposition of the tensor.

Let us consider an isotropic granular material whose elastic properties are characterized by the bulk compression modulus k and the shear modulus μ. The set F is approximated by a Mises cylinder:

$$F = \{\sigma \mid \tau(\sigma) \leqslant \tau_s\}.$$

Here $\tau(\sigma) = \sqrt{\sigma' : \sigma'/2}$ is the shearing stress rate, the quantity with a prime is the tensor deviator, and τ_s is the yield point of the granules. For description of admissible stresses, we use a Mises–Schleicher circular cone

$$K = \{\sigma \mid \tau(\sigma) \leqslant æp(\sigma)\}.$$

Here $p(\sigma)$ is the hydrostatic pressure and $æ$ is the internal friction coefficient. Equations (1.3), expressing the stress tensor σ in terms of the conditional stress tensor s in an elastic material, become

$$\sigma' = s' - \lambda\mu\sigma'/(æp(\sigma)), \qquad p(\sigma) = p(s) + \lambda æ k, \qquad \lambda(\tau(\sigma) - æp(\sigma)) = 0.$$

Three cases are therefore possible. If $\tau(s) \leqslant æp(s)$, then $\sigma = s$. If $\tau(s) > æp(s)$ and $\mu p(s) + æk\tau(s) \leqslant 0$, then $\sigma = 0$. In this case, the projection s is the vertex of the cone K. If $\tau(s) > æp(s)$ and $\mu p(s) + æk\tau(s) > 0$, the projection s belongs to the conical surface and is determined from the formulas

$$\sigma' = \frac{æp(\sigma)}{\tau(s)}\, s', \qquad p(\sigma) = \frac{\mu p(s) + æk\tau(s)}{\mu + æ^2 k}. \tag{1.6}$$

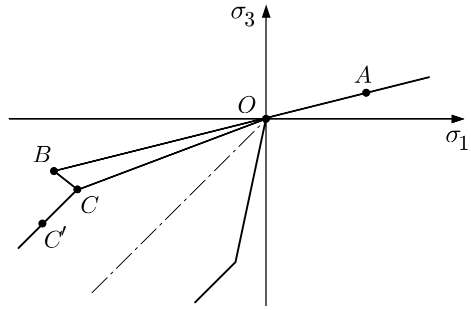

Fig. 2. Trajectories of the stresses s and σ.

2. Elastoplastic Waves. We consider a plane longitudinal shock wave propagating in an infinite bulk of a granular material in the x_1 direction of Cartesian coordinates. The wave amplitude is considered small enough. Then, the following simple dynamic and kinematic compatibility equations are valid at the shock front [14]:

$$\rho c[v_1] = -[\sigma_1], \qquad c[\varepsilon_1] = -[v_1]. \tag{2.1}$$

Here c is the wave propagation velocity, v_1 is particle mass velocity, and σ_1, $\sigma_2 = \sigma_3$, and ε_1 are the nonzero components of the stress and strain tensors; square brackets denote a jump of the function at the discontinuity. For closure of system (2.1), it should be supplemented by an equation linking σ_1 and ε_1. As a result, the shock adiabat equation [the curve of admissible shock-wave transitions from a fixed state (ε_1^0, v_1^0) ahead of the shock front to the state (ε_1, v_1) behind the front] takes the form

$$\rho(v_1 - v_1^0)^2 = (\sigma_1 - \sigma_1^0)(\varepsilon_1 - \varepsilon_1^0). \tag{2.2}$$

The solution of the equation $\sigma_1 = \sigma_1(\varepsilon_1)$ depends on the sign and nature of the deformation (elastic or elastoplastic loading and unloading).

Treating ε_1 as a parameter, we construct the trajectories of the stresses $s(\varepsilon_1)$ and $\sigma(\varepsilon_1)$ on the plane (σ_1, σ_3). The case where the internal friction coefficient is in the range $[2\mu/(\sqrt{3}k), \sqrt{3}/2]$ is considered in [10]. In the present paper, we examine the more interesting case: $æ < 2\mu/(\sqrt{3}k)$ and $æ \leqslant \sqrt{3}/2$. The position of the uniaxial elastic deformation line AB relative to the Mises–Schleicher cone and the Mises cylinder for this case is shown in Fig. 2. Under axial compression, the material with such characteristics is always in the limiting sate.

For loosening of the material ($\varepsilon_1 > 0$), the conditional stresses s_1 and s_3 are given by

$$s_1 = (k + 4\mu/3)\varepsilon_1, \qquad s_3 = (k - 2\mu/3)\varepsilon_1, \tag{2.3}$$

and the true stresses σ_1 and σ_3 (the projections of conditional stresses onto the cone) are equal to zero. For compression in the range $\varepsilon_1^B \leqslant \varepsilon_1 \leqslant 0$, where $\varepsilon_1^B = -\tau_s/(æh)$, the trajectory of conditional stresses is still defined by Eqs. (2.3) and the true stresses, by virtue of (1.6) have the form

$$\sigma_1 = \Big(1 + \frac{2æ}{\sqrt{3}}\Big)h\varepsilon_1, \qquad \sigma_3 = \Big(1 - \frac{æ}{\sqrt{3}}\Big)h\varepsilon_1, \qquad h = \frac{1 + 2æ/\sqrt{3}}{1 + æ^2 k/\mu}\, k.$$

Transition to the plastic state corresponds to the point B in Fig. 2. At this point, the material ceases to resist shear, and additional compaction is required to restore the bearing strength. Such compaction corresponds to the segment BC under unchanged stresses with a jump of the strain

$$\Delta\varepsilon_1 = \varepsilon_1^B - \varepsilon_1^C = \frac{2/\sqrt{3} - æk/\mu}{1 + 2æ/\sqrt{3}}\,\frac{\tau_s}{k},$$

where $\varepsilon_1^C = -\tau_s/(æk)$. The parametric equations of the straight line BC have the form

$$s_1 = \Big(k - \frac{2\mu}{\sqrt{3}æ}\Big)\varepsilon_1 - \frac{2\tau_s}{\sqrt{3}}\Big(1 + \frac{\mu}{æ^2 k}\Big), \qquad s_3 = \Big(k + \frac{\mu}{\sqrt{3}æ}\Big)\varepsilon_1 + \frac{\tau_s}{\sqrt{3}}\Big(1 + \frac{\mu}{æ^2 k}\Big).$$

The interval $\varepsilon_1 < \varepsilon_1^C$ corresponds to plastic compression (CC' ray). In this interval, the stresses $\sigma = s$ are defined by the formulas

$$\sigma_1 = s_1 = k\varepsilon_1 - 2\tau_s/\sqrt{3}, \qquad \sigma_3 = s_3 = k\varepsilon_1 + \tau_s/\sqrt{3}.$$

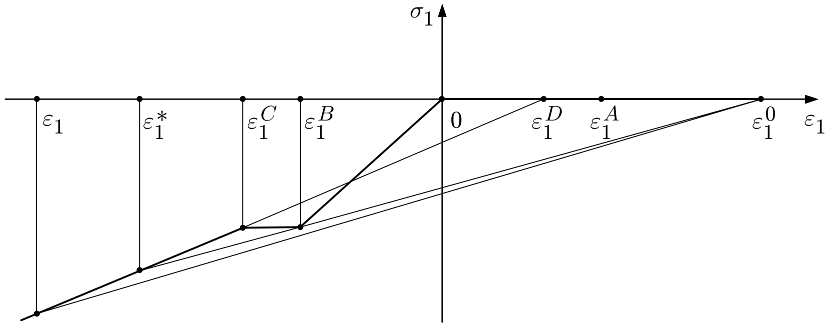

Fig. 3. Dependence of σ_1 on ε_1.

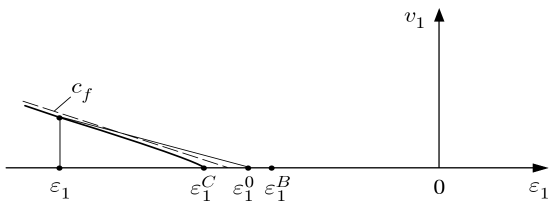

Fig. 4. Shock adiabat for $\varepsilon_1^C < \varepsilon_1^0 < \varepsilon_1^B$.

Thus, the trajectory of $s(\varepsilon_1)$ is the broken line $ABCC'$ and the trajectory of $\sigma(\varepsilon_1)$ is the broken line OCC'. A plot of the function $\sigma_1(\varepsilon_1)$ is given in Fig. 3.

Let us consider only waves that move in the positive direction of the x_1 axis, for which $c > 0$. Assuming that $v_1^0 = 0$, we construct the shock adiabats of compression waves ($\varepsilon_1 < \varepsilon_1^0$).
If ahead of the shock front, the material is compressed to a plastic state ($\varepsilon_1^0 \leqslant \varepsilon_1^C$), a plastic shock wave characteristic of an ordinary elastoplastic material occurs in the material [15]. Equation (2.2) becomes

$$v_1 = -c_f(\varepsilon_1 - \varepsilon_1^0),$$

where $c_f = \sqrt{k/\rho}$ is the velocity of plastic shock waves. On the plane (ε_1, v_1), the shock adiabat has the form a ray that issues from the point ε_1^0. For $\varepsilon_1^C < \varepsilon_1^0 < \varepsilon_1^B$, the shock adiabat is described by the hyperbola equation that follow from (2.2):

$$(v_1/c_f)^2 = (\varepsilon_1 - \varepsilon_1^C)(\varepsilon_1 - \varepsilon_1^0). \tag{2.4}$$

This wave arises only if $\varepsilon_1 < \varepsilon_1^C$ (Fig. 4). This is a compression wave of the material that lost the bearing strength at the moment the granules enters a plastic state. Its velocity is evaluated from the formula

$$c = c_f\sqrt{(\varepsilon_1 - \varepsilon_1^C)/(\varepsilon_1 - \varepsilon_1^0)}. \tag{2.5}$$

If $\varepsilon_1^B \leqslant \varepsilon_1^0 \leqslant 0$, then one or two waves (Fig. 5) propagate, depending on the degree of compression. For intense compression with $\varepsilon_1 < \varepsilon_1^C$, a two-wave configuration of discontinuities arises: an elastic precursor in a loose medium moving at the velocity

$$c_p = \sqrt{(1 + 2æ/\sqrt{3})h/\rho},$$

and a plastic compression wave, whose velocity is determined from formula (2.5) after replacing ε_1^0 by ε_1^B. The elastic precursor transforms the material from the state ε_1^0 to the limiting elastic state ε_1^B. Its shock adiabat is linear. The shock adiabat of the plastic compression wave is obtained by parallel translation of hyperbola (2.4) by the magnitude v_1^C along the v_1 axis. For weak compression, where $\varepsilon_1 \geqslant \varepsilon_1^B$, a plastic compression wave does not arise. In the interval $\varepsilon_1^C \leqslant \varepsilon_1 < \varepsilon_1^B$, the bearing strength of the material behind the elastic precursor front is not able to restore; therefore, the plastic wave turns into a fixed discontinuity.

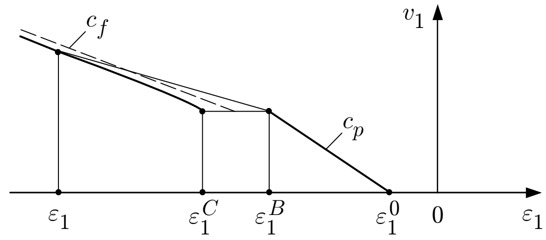

Fig. 5. Shock adiabat for $\varepsilon_1^B \leqslant \varepsilon_1^0 \leqslant 0$.

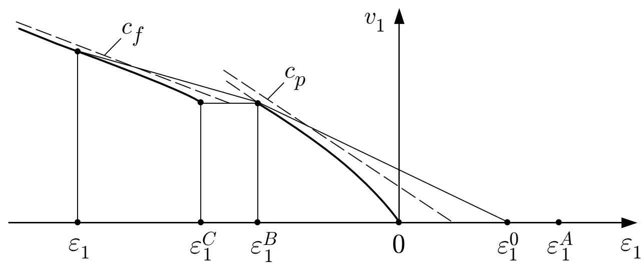

Fig. 6. Shock adiabat for $0 < \varepsilon_1^0 \leqslant \varepsilon_1^A$.

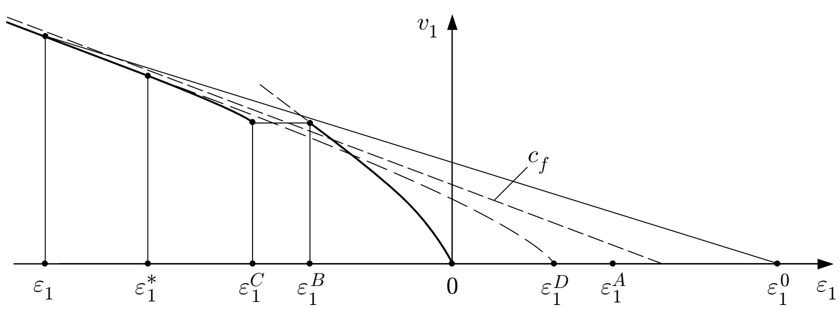

Fig. 7. Shock adiabat for $\varepsilon_1^0 > \varepsilon_1^A$.

If the material is loose ($\varepsilon_1^0 > 0$) ahead of the shock front, a solitary elastic signoton

$$(v_1/c_p)^2 = \varepsilon_1(\varepsilon_1 - \varepsilon_1^0) \tag{2.6}$$

or a two-wave configuration with an elastic precursor signoton (2.6) with a plastic compression wave (2.4) are possible (Fig. 6). Following the adopted nomenclature [6], a signoton is a shock wave whose passage leads to an instantaneous change in the sign of deformation. Plots of the shock adiabats (2.6) and (2.4) are branches of hyperbolas whose asymptotes are inclined to the abscissa at angles arctan c_p and arctan c_f, respectively. The asymptote of the first adiabat issues from the point $\varepsilon_1^0/2$. The asymptote of the second adiabat issues from the point that lies in the middle between ε_1^C and ε_1^B. Such a pattern is observed only for rather low loosening: $\varepsilon_1^0 \leqslant \varepsilon_1^A$. An excess over the critical value of $\varepsilon_1^A = \Delta\varepsilon_1 + \varepsilon_1^D$ [$\varepsilon_1^D = 2\tau_s/(\sqrt{3}k)$] leads to overturning of the intense compression waves because the velocity of the precursor signoton becomes lower than c_f. For $\varepsilon_1^0 > \varepsilon_1^A$ (Fig. 7), the shock adiabat consists of three branches: the adiabat of elastic signotons (2.6), the adiabat of plastic compression waves (2.4), and the adiabat of plastic signotons

$$(v_1/c_f)^2 = (\varepsilon_1 - \varepsilon_1^D)(\varepsilon_1 - \varepsilon_1^0).$$

The system of waves for a specified value of ε_1 is easy to determine from Fig. 3, taking into account that the quantity ρc^2 is equal to the angular coefficient of the secant on the plane $(\varepsilon_1, \sigma_1)$. From Fig. 3 it follows that for $\varepsilon_1^B \leqslant \varepsilon_1 < 0$, a solitary elastic signoton forms. If $\varepsilon_1^* < \varepsilon_1 < \varepsilon_1^C$, where

$$\varepsilon_1^* = \frac{\varepsilon_1^0 + (2æ/\sqrt{3})\varepsilon_1^B}{\varepsilon_1^0 - \varepsilon_1^A}\,\varepsilon_1^C$$

is the strain at the point where the ray passing from the initial state to the state B intersects with the plot of the function $\sigma_1(\varepsilon_1)$, then the elastic signoton is followed by a plastic compression wave. For strain $\varepsilon_1 = \varepsilon_1^*$, the velocities of these waves become equal. For $\varepsilon_1 < \varepsilon_1^*$, the shock-wave transition is described by a solitary plastic signoton.

Unlike in [10], in the present study, we consider a more complex version of specification of material parameters, which determines a model in which plastic compression waves form. An even more complex pattern arises in the case $æ > \sqrt{3}/2$, where the material can render resistance to uniaxial tension, exhibiting elastic and plastic properties. This case requires an additional investigation.

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